Sisteme de Coordonate

```
Clear["Global`*"]
```

Mathematica ofera posibilitatea tratarii simbolice a analizei vectoriale

```
Needs["VectorAnalysis`"]
```

Tipuri de coordonate:

Mathematica va ofera posibilitatea sa constatati ce contine acest pachet

```
?Calculus`VectorAnalysis`*
```

Sa vizualizam tipurile de coordonate apelabile prin acest pachet

CoordinateSystem da numele tipului de coordonate implicit

```
?Coordinates
```

```
Coordinates[ ] gives a list of the default coordinate variables in the
  default coordinate system. Coordinates[coordsys] gives a list of
  the default coordinate variables in the coordinate system coordsys.
```

Ne punem acum problema alegerii sistemului de coordonate si determinarea limitelor acestora:

```
CoordinateSystem
Coordinates[]
```

```
Cartesian
```

```
{Xx, Yy, Zz}
```

```
SetCoordinates[Cartesian[x, y, z]];
CoordinateRanges[Cartesian]
```

```
{-∞ < x < ∞, -∞ < y < ∞, -∞ < z < ∞}
```

```
SetCoordinates[Spherical[ρ, θ, ϕ]];
CoordinateRanges[Spherical]
```

$\{0 \le \rho < \infty, 0 \le \theta \le \pi, -\pi < \phi \le \pi\}$

```
SetCoordinates[Cylindrical[r, θ, z]];
CoordinateRanges[Cylindrical]
```

$\{0 \le r < \infty, -\pi < \theta \le \pi, -\infty < z < \infty\}$

CoordinatesToCartesian da relatiile de trecere de la un sistem cartezian de coordonate la cele curbilinii ortogonale.

```
SetCoordinates[Spherical[ρ, θ, ϕ]];
CoordinatesToCartesian[{ρ, θ, ϕ}]
```

{ρ Cos[ϕ] Sin[θ], ρ Sin[θ] Sin[ϕ], ρ Cos[θ]}

```
SetCoordinates[Cylindrical[r, θ, z]];
CoordinatesToCartesian[{r, θ, z}]
```

{r Cos[θ], r Sin[θ], z}

CoordinatesFromCartesian da relatiile de transformare ale coordonatelor curbilinii in functie de coordonatele carteziene

```
CoordinatesToCartesian[{ρ, θ, ϕ}, Spherical]
```

{ρ Cos[ϕ] Sin[θ], ρ Sin[θ] Sin[ϕ], ρ Cos[θ]}

```
CoordinatesFromCartesian[{x, y, z}, Spherical]
```

$$\left\{\sqrt{x^2+y^2+z^2}, \mathrm{ArcCos}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right], \mathrm{ArcTan}[x, y]\right\}$$

```
CoordinatesToCartesian[{r, θ, z}, Cylindrical]
```

```
{r Cos[θ], r Sin[θ], z}
```

```
CoordinatesFromCartesian[{x, y, z}, Cylindrical]
```

$\left\{\sqrt{x^2 + y^2}, \text{ArcTan}[x, y], z\right\}$

Exemplu:

Sa exprimam funxtia f(x,y,z)=x^2+y^2+z^2 in coordonate curbilinii ortogonale (ρ,θ,ϕ) si (ρ,θ,z).

```
Clear[ρ, θ, ϕ, r, f];
```

```
f[x_, y_, z_] := x^2 * y^2 * z^2
```

```
CoordinatesToCartesian[{ρ, φ, θ}, Spherical]
```

```
{ρ Cos[θ] Sin[φ], ρ Sin[θ] Sin[φ], ρ Cos[φ]}
```

```
f[x, y, z] /. {x → %[[1]], y → %[[2]], z → %[[3]]} // Simplify
```

$\rho^6 \text{Cos}[\theta]^2 \text{Cos}[\varphi]^2 \text{Sin}[\theta]^2 \text{Sin}[\varphi]^4$

```
CoordinatesToCartesian[{r, θ, z}, Cylindrical]
```

```
{r Cos[θ], r Sin[θ], z}
```

```
f[x, y, z] /. {x → %[[1]], y → %[[2]], z → %[[3]]} // Simplify
```

$r^4 z^2 \text{Cos}[\theta]^2 \text{Sin}[\theta]^2$

Operatii vectoriale:

Produsul scalar al vectorilor

```
SetCoordinates[Cartesian[x, y, z]];
vec1 = {a1, b1, c1}
vec2 = {a2, b2, c2}
vec3 = {a3, b3, c3}
DotProduct[vec1, vec2]
vec1.vec2
```

```
{a1, b1, c1}
```

```
{a2, b2, c2}
```

```
{a3, b3, c3}
```

```
a1 a2 + b1 b2 + c1 c2
```

```
a1 a2 + b1 b2 + c1 c2
```

Produsul vectorial

```
CrossProduct[vec1, vec2]
```

```
{-b2 c1 + b1 c2, a2 c1 - a1 c2, -a2 b1 + a1 b2}
```

Triplu produs csalar

```
ScalarTripleProduct[vec1, vec2, vec3]
```

```
-a3 b2 c1 + a2 b3 c1 + a3 b1 c2 - a1 b3 c2 - a2 b1 c3 + a1 b2 c3
```

Operatorii Grad si Laplacian in coord. curbilinii:

```
Clear[f];
```

```
Grad[f[x, y, z], Cartesian]
```

$\{f^{(1,0,0)}[x, y, z], f^{(0,1,0)}[x, y, z], f^{(0,0,1)}[x, y, z]\}$

```
Grad[f[ρ, θ, ϕ], Spherical]
```

$\left\{f^{(1,0,0)}[\rho, \theta, \phi], \frac{f^{(0,1,0)}[\rho, \theta, \phi]}{\rho}, \frac{\mathrm{Csc}[\theta] f^{(0,0,1)}[\rho, \theta, \phi]}{\rho}\right\}$

```
Grad[f[r, θ, ϕ], Cylindrical]
```

$\left\{f^{(1,0,0)}[r, \theta, \phi], \frac{f^{(0,1,0)}[r, \theta, \phi]}{r}, 0\right\}$

```
Laplacian[f[x, y, z], Cartesian]
```

$f^{(0,0,2)}[x, y, z] + f^{(0,2,0)}[x, y, z] + f^{(2,0,0)}[x, y, z]$

```
Laplacian[f[ρ, θ, ϕ], Spherical]
```

$$\frac{1}{\rho^2}\left(\mathrm{Csc}[\theta]\left(\mathrm{Csc}[\theta] f^{(0,0,2)}[\rho, \theta, \phi] + \mathrm{Cos}[\theta] f^{(0,1,0)}[\rho, \theta, \phi] + \mathrm{Sin}[\theta] f^{(0,2,0)}[\rho, \theta, \phi] + 2\rho\, \mathrm{Sin}[\theta] f^{(1,0,0)}[\rho, \theta, \phi] + \rho^2\, \mathrm{Sin}[\theta] f^{(2,0,0)}[\rho, \theta, \phi]\right)\right)$$

```
Laplacian[f[r, θ, z], Cylindrical]
```

$$\frac{1}{r}\left(r\, f^{(0,0,2)}[r, \theta, z] + \frac{f^{(0,2,0)}[r, \theta, z]}{r} + f^{(1,0,0)}[r, \theta, z] + r\, f^{(2,0,0)}[r, \theta, z]\right)$$

Exemplu:

Sa determinam Grad si laplacianul din f(x,y,z)=x^2+y^2+z^2 in coordonate curbilinii ortogonale (ρ,θ,ϕ) si (ρ,θ,z).

```
Clear[ρ, θ, ϕ, r, f];
```

```
f[x_, y_, z_] := x^2 + y^2 + z^2
```

```
Grad[f[x, y, z], Cartesian]
```

{2 x, 2 y, 2 z}

```
Laplacian[f[x, y, z], Cartesian]
```

6

```
CoordinatesToCartesian[{ρ, φ, θ}, Spherical]
```

{ρ Cos[θ] Sin[φ], ρ Sin[θ] Sin[φ], ρ Cos[φ]}

```
g[ρ, θ, ϕ] =
 f[x, y, z] /. {x → %[[1]], y → %[[2]], z → %[[3]]} // Simplify
```

ρ^2

```
Grad[g[ρ, θ, ϕ], Spherical]
```

{2 ρ, 0, 0}

```
Laplacian[g[ρ, θ, ϕ], Spherical]
```

```
6
```

```
CoordinatesToCartesian[{r, θ, z}, Cylindrical]
```

```
{r Cos[θ], r Sin[θ], z}
```

```
h[r, θ, z] =
 f[x, y, z] /. {x → %[[1]], y → %[[2]], z → %[[3]]} // Simplify
```

$r^2 + z^2$

```
Grad[h[r, θ, z], Cylindrical]
```

```
{2 r, 0, 2 z}
```

```
Laplacian[h[r, θ, z], Cylindrical]
```

```
6
```

Operatorii Divsi Rot(Curl) in coord. curbilinii:

Div fiind un scalar si reprezentand un flux, va trebui sa explicitam componentele vectorului sursa:

```
Div[{x^2, y^2, z^2}, Cartesian]
```

```
2 x + 2 y + 2 z
```

```
Div[{ρ^2 * Sin[2 * θ], ϕ^3 * Cos[θ^2], ρ * Sqrt[ϕ^2]}, Spherical]
```

$$\frac{1}{\rho^2}\left(\mathrm{Csc}[\theta]\left(\frac{\rho^2\,\phi}{\sqrt{\phi^2}}+\rho\,\phi^3\,\mathrm{Cos}[\theta]\,\mathrm{Cos}\left[\theta^2\right]+4\,\rho^3\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[2\,\theta]-2\,\theta\,\rho\,\phi^3\,\mathrm{Sin}[\theta]\,\mathrm{Sin}\left[\theta^2\right]\right)\right)$$

```
Div[{r^2 * Sin[2 * θ], z^3 * Cos[θ^2], r * Sqrt[z^2]}, Cylindrical]
```

$$\frac{\frac{r^2\,z}{\sqrt{z^2}}+3\,r^2\,\mathrm{Sin}[2\,\theta]-2\,z^3\,\theta\,\mathrm{Sin}\left[\theta^2\right]}{r}$$

Apelarea Rotaionalului implica precizarea componentelor expresiei vectoriale

```
Curl[{x, y, z}, Cartesian]
```

{0, 0, 0}

```
Curl[{ρ * Sin[θ], Cos[θ] * ϕ, ϕ^2}, Spherical]
```

$$\left\{\frac{\left(-\rho\,\mathrm{Cos}[\theta]+\rho\,\phi^2\,\mathrm{Cos}[\theta]\right)\mathrm{Csc}[\theta]}{\rho^2},\ -\frac{\phi^2}{\rho},\ \frac{-\rho\,\mathrm{Cos}[\theta]+\phi\,\mathrm{Cos}[\theta]}{\rho}\right\}$$

```
Curl[{r * Sin[θ], Cos[θ] * z, z^2 + 1}, Cylindrical]
```

$$\left\{-\mathrm{Cos}[\theta],\ 0,\ \frac{-r\,\mathrm{Cos}[\theta]+z\,\mathrm{Cos}[\theta]}{r}\right\}$$

Rot si Div in alte sisteme de coordonate

```
SetCoordinates[Spherical[ρ, ϕ, θ]]
```

```
Spherical[ρ, ϕ, θ]
```

```
Curl[CoordinatesFromCartesian[F[{x, y, z}]]]
```

$$\left\{\frac{1}{\rho}\,(\mathrm{ArcTan}[P[x,y,z],Q[x,y,z]]\,\mathrm{Cot}[\phi]),\ -\frac{\mathrm{ArcTan}[P[x,y,z],Q[x,y,z]]}{\rho},\ \frac{1}{\rho}\,\mathrm{ArcCos}\left[R[x,y,z]\Big/\left(\sqrt{\left(P[x,y,z]^2+Q[x,y,z]^2+R[x,y,z]^2\right)}\right)\right]\right\}$$

```
Div[CoordinatesFromCartesian[F[{x, y, z}]]]
```

$$\frac{1}{\rho^2}\left(\mathrm{Csc}[\phi]\left(\rho\,\mathrm{ArcCos}\left[R[x,y,z]\Big/\left(\sqrt{\left(P[x,y,z]^2+Q[x,y,z]^2+R[x,y,z]^2\right)}\right)\right]\mathrm{Cos}[\phi]+2\,\rho\,\sqrt{\left(P[x,y,z]^2+Q[x,y,z]^2+R[x,y,z]^2\right)}\,\mathrm{Sin}[\phi]\right)\right)$$

Exemplu:

Sa calculam $\nabla.\left(f\,(r)\,\vec{r}\right)$ in coordonate carteziene.

```
Div[f[r1] * {x, y, z}]
```

$$3\,f\left[\sqrt{x^2+y^2+z^2}\right]+\frac{x^2\,f'\left[\sqrt{x^2+y^2+z^2}\right]}{\sqrt{x^2+y^2+z^2}}+\frac{y^2\,f'\left[\sqrt{x^2+y^2+z^2}\right]}{\sqrt{x^2+y^2+z^2}}+\frac{z^2\,f'\left[\sqrt{x^2+y^2+z^2}\right]}{\sqrt{x^2+y^2+z^2}}$$

```
Clear[r];
SetCoordinates[Cartesian[x, y, z]];
r1 := Sqrt[x^2 + y^2 + z^2]

Div[f[r1] {x, y, z}] // FullSimplify
```

$$3\, f\left[\sqrt{x^2+y^2+z^2}\right] + \sqrt{x^2+y^2+z^2}\; f'\left[\sqrt{x^2+y^2+z^2}\right]$$

Sa precizam ca $\vec{r} = r\,\vec{e}_r = r\,(1, 0,0)$.

```
Clear[r]
SetCoordinates[Spherical[r, θ, ϕ]];
Div[f[r] * r {1, 0, 0}]
```

$$\frac{1}{r^2}\left(\mathrm{Csc}[\theta]\left(3\, r^2\, f[r]\, \mathrm{Sin}[\theta] + r^3\, \mathrm{Sin}[\theta]\, f'[r]\right)\right)$$

```
% // FullSimplify
```

$$3\, f[r] + r\, f'[r]$$

```
Clear[r]
SetCoordinates[Cylindrical[r, θ, z]];
Div[f[r] * r {1, 0, 0}]
```

$$\frac{2\, r\, f[r] + r^2\, f'[r]}{r}$$

```
% // FullSimplify
```

$$2\, f[r] + r\, f'[r]$$

Exemplu:

Sa calculam $\nabla \times \left(f\,(r)\; x \star \vec{r}\right)$ in coordonate carteziene.

```
Curl[f[r] * x * {x, y, z}, Cartesian]
```

```
{0, -z f[r], y f[r]}
```

```
Clear[r];
SetCoordinates[Cartesian[x, y, z]];
r1 := Sqrt[x^2 + y^2 + z^2]

Curl[f[r1] * x * {x, y, z}] // FullSimplify
```

$\left\{0, -z\, f\left[\sqrt{x^2+y^2+z^2}\right], y\, f\left[\sqrt{x^2+y^2+z^2}\right]\right\}$

Exemplu general pentru operatorii Grad, Laplacian, Div si Rot(Curl) in coord. curbilinii:

```
SetCoordinates[Cartesian[x, y, z]];
Grad[x^2]
Laplacian[x^2]
Laplacian[f[x, y, z]]
```

```
{2 x, 0, 0}
```

```
2
```

```
6
```

```
SetCoordinates[Spherical[r, θ, ϕ]];
Grad[r^2]
Div[r {1, 0, 0}]
Curl[r {1, 0, 0}]
Curl[f[r] {1, 0, 0}]
Laplacian[f[r, ϑ, φ]] // FullSimplify
```

{2 r, 0, 0}

3

{0, 0, 0}

{0, 0, 0}

6

```
SetCoordinates[Cylindrical[r, ϑ, z]];
Grad[r^2]
Laplacian[f[r, ϑ, z]] // Simplify
```

{2 r, 0, 0}

$6 + \frac{2}{r^2}$

```
SetCoordinates[Cartesian[x, y, z]];
r := Sqrt[x^2 + y^2 + z^2] ;
Grad[1 / r]
Div[Grad[1 / r]]
Laplacian[1 / r]
```

$$\left\{-\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}},\ -\frac{y}{\left(x^2+y^2+z^2\right)^{3/2}},\ -\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}\right\}$$

$$\frac{3\,x^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{3\,y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{3\,z^2}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{3}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{3\,x^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{3\,y^2}{\left(x^2+y^2+z^2\right)^{5/2}}+\frac{3\,z^2}{\left(x^2+y^2+z^2\right)^{5/2}}-\frac{3}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Cazul unidimensional:

Exemplificare pentru operatorul Δ (coord. carteziene si sferice)

```
SetCoordinates[Cartesian[x, y, z]];
Laplacian[f[x]] == 0
```

$f''[x] == 0$

```
DSolve[Laplacian[f[x]] == 0, f[x], x] // Flatten
```

$\{f[x] \to C[1] + x\, C[2]\}$

```
SetCoordinates[Spherical[ρ, ϑ, ϕ]];
Laplacian[f[ρ]] == 0 // Simplify
```

$$\frac{2\, f'[\rho]}{\rho} + f''[\rho] == 0$$

```
DSolve[Laplacian[f[ρ]] == 0, f[ρ], ρ] // Flatten
```

$$\left\{f[\rho] \to -\frac{C[1]}{\rho} + C[2]\right\}$$

Elemente utile pentru integrare:

Determinarea coeficientilor Lame

```
SetCoordinates[Spherical[ρ, θ, ϕ]];
ScaleFactors[]
```

```
{1, ρ, ρ Sin[θ]}
```

Stabilirea formei determinantului matricii transformarii de la sistemul implicit de coordonate in cel cartezian (Jacobianul transformarii)

```
JacobianDeterminant[]
```

$\rho^2 \mathrm{Sin}[\theta]$

Matricea transformarii

```
JacobianMatrix[]
```

```
{{Cos[ϕ] Sin[θ], ρ Cos[θ] Cos[ϕ], -ρ Sin[θ] Sin[ϕ]},
 {Sin[θ] Sin[ϕ], ρ Cos[θ] Sin[ϕ], ρ Cos[ϕ] Sin[θ]}, {Cos[θ], -ρ Sin[θ], 0}}
```

Un exemplu tipic de integrala tripla $\int_0^1 \int_0^\pi \int_0^{2\pi} \frac{1}{r^2} \rho^2 \sin\theta \, dr \, d\theta \, d\phi$

```
Integrate[ρ^-2 * JacobianDeterminant[], {ρ, 0, 1}, {θ, 0, Pi}, {ϕ, 0, 2 Pi}]
```

4π

```
Clear[r]
```

```
SetCoordinates[Cylindrical[r, θ, z]];
ScaleFactors[];
JacobianDeterminant[]
```

```
r
```

```
Integrate[r^-2 * JacobianDeterminant[], {r, 0, R}, {θ, 0, Pi}, {z, 0, L}]
```

```
L π Log[R]
```

Identitati operatoriale:

∇×[∇f(x,y,z)]=0

```
Clear[f]
```

```
Curl[Grad[f[x, y, z]]]
```

```
{0, 0, 0}
```

∇·[∇×f(x,y,z)]=0

```
Clear[F, P, Q, R]
```

```
F[{x_, y_, z_}] := {P[x, y, z], Q[x, y, z], R[x, y, z]}
```

```
Div[Curl[F[{x, y, z}]]]
```

```
0
```

Exercitii:

Raspunsuri:

exercitiul 1

```
Needs["VectorAnalysis`"]
SetCoordinates[Cartesian[x, y, z]];
f[x_, y_] := x^2 + y^2
```

```
Grad[f[x, y], Cartesian]
```

```
{2 x, 2 y, 0}
```

exercitiul 2

2 exercitiul

```
SetCoordinates[Spherical[ρ, ϕ, θ]];
f[ρ_, θ_] := ρ^2 Sin[θ]
Grad[f[ρ, θ], Spherical ]
```

```
{2 ρ Sin[θ], 0, ρ Cos[θ] Csc[ϕ]}
```

exercitiul 3

```
SetCoordinates[Spherical[ρ, ϕ, θ]];
f[ρ_, ϕ_, θ_] := ρ^2 - 2 ρ Sin[θ] + ϕ^2
Grad[f[ρ, ϕ, θ], Spherical ]
```

$$\left\{2\rho - 2\,\mathrm{Sin}[\theta],\ \frac{2\phi}{\rho},\ -2\,\mathrm{Cos}[\theta]\,\mathrm{Csc}[\phi]\right\}$$

exercitiul 4

```
SetCoordinates[Spherical[ρ, ϕ, θ]];
f[ρ_, ϕ_, θ_] := ρ^2 Cos[θ^2]
    Laplacian[f[ρ, ϕ, θ], Spherical ] // Simplify
```

$\text{Cos}\left[\theta^2\right]\left(6-4\,\theta^2\,\text{Csc}[\phi]^2\right)-2\,\text{Csc}[\phi]^2\,\text{Sin}\left[\theta^2\right]$

exercitiul 5

```
SetCoordinates[Cartesian[x, y, z]];
f[x_, y_, z_] := x^3 - 2 y^2 - z
```

```
Laplacian[f[x, y, z], Cartesian ] // Simplify
```

$-4+6\,x$

exercitiul 6

```
SetCoordinates[Cartesian[x, y, z]];
```

```
Div[{x, y^2, z}, Cartesian]
```

$2+2\,y$

exercitiul 7

```
SetCoordinates[Spherical[ρ, ϕ, θ]];
```

```
Div[{ρ^2 Θ, ρ^2 Sin[θ] - ρ^2, ρ}, Spherical] // Simplify
```

```
ρ (4 Θ + Cot[ϕ] (-1 + Sin[θ]))
```

exercitiul 8

```
SetCoordinates[Cartesian[x, y, z]];
```

```
Curl[{y, -x, x}, Cartesian]
```

```
{0, -1, -2}
```

exercitiul 9

```
SetCoordinates[Cartesian[x, y, z]];
Curl[{Sin[x], Cos[y], x}, Cartesian]
```

```
{0, -1, 0}
```

exercitiul 10

```
SetCoordinates[Cartesian[x, y, z]];
```

$$\text{Curl}\left[\left\{\frac{y^2}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right\}, \text{Cartesian}\right] \text{ // Simplify}$$

$$\left\{0, 0, \frac{y^2 - x^2 (1 + 2 y)}{\left(x^2+y^2\right)^2}\right\}$$

```
SetCoordinates[Spherical[ρ, ϕ, θ]];
```

```
Curl[CoordinatesFromCartesian[{x, y, z}]]
```

$$\left\{\frac{\text{ArcTan}[x, y]\ \text{Cot}[\phi]}{\rho},\ -\frac{\text{ArcTan}[x, y]}{\rho},\ \frac{\text{ArcCos}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right]}{\rho}\right\}$$

```
SetCoordinates[Cartesian[x, y, z]];
```

$$\text{Curl}\left[\left\{\frac{y^2}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 0\right\},\ \text{Cartesian}\right]\ //\ \text{Simplify}$$

$$\left\{0,\ 0,\ \frac{y^2 - x^2\ (1 + 2\ y)}{\left(x^2+y^2\right)^2}\right\}$$

```
SetCoordinates[Cylindrical[r, θ, z]];
```

```
Curl[CoordinatesFromCartesian[{x, y, z}]]
```

$$\left\{0,\ 0,\ \frac{\text{ArcTan}[x, y]}{r}\right\}$$